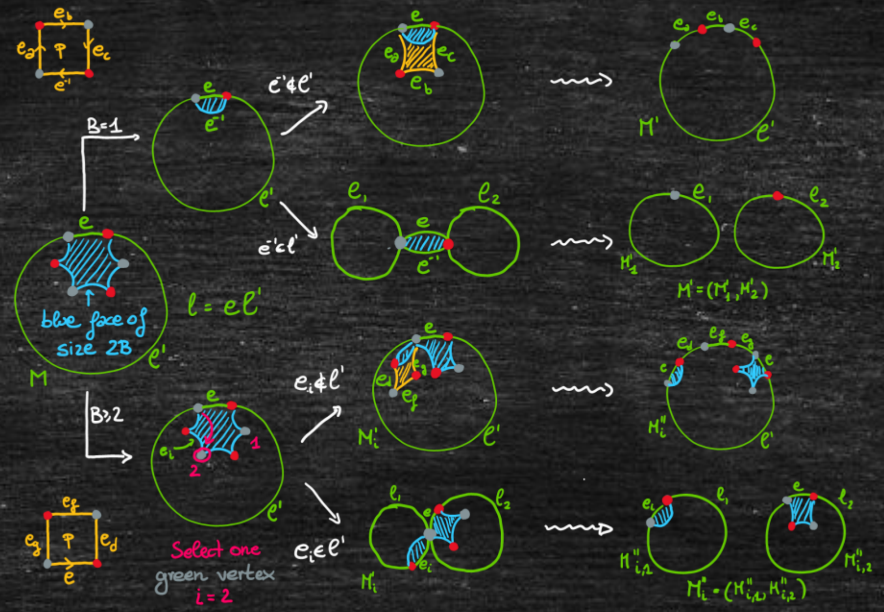

Lattice Yang-Mills theory in the large N limit via Random Surfaces

(ongoing joint work with Jasper Shogren-Knaak & Sky Cao)

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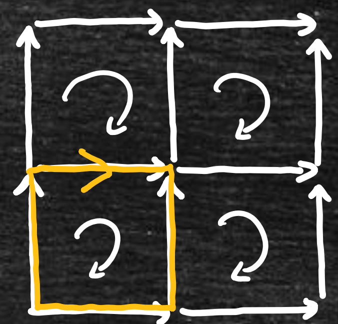
IMSI, Two-Dimensional Random Geometry, July 9, 2024

Lattice Yang-Mills: the model

Here $\mathcal{P}_\Lambda$ contains 8 plaquettes (4 pos. oriented, 4 neg. oriented)



$\Lambda \subseteq \mathbb{Z}^2$

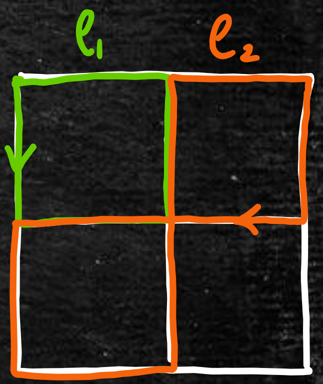
- $d \geq 2$, $N \geq 1$, $\beta \in \mathbb{R}_+$
- We work on $\Lambda \subseteq \mathbb{Z}^d$ finite.
- E_Λ^+ = Positively oriented edges of Λ
- $\mathcal{P}_\Lambda$ is the collection of plaquettes in Λ
- $U(N)$ = Group of $N \times N$ complex-valued matrices s.t. $U^* U = U U^* = \mathbb{I}$ (conjugate transpose; $\mathbb{I}$ ↳ identity.)

  ↳ All our results will probably extend to other groups of matrices.
- $\mathrm{Tr}(\cdot)$ = trace, $\mathrm{tr}(\cdot) = \frac{1}{N}\mathrm{Tr}(\cdot)$
- dQ_e is the Haar measure on $U(N)$.
- $Q = (Q_e)_{e \in E_\Lambda^+}$, $Q_p = Q_{e_1} Q_{e_2} Q_{e_3} Q_{e_4}$ when $p = (e_1, e_2, e_3, e_4)$ $\quad [Q_{e^{-1}} = Q_e^{-1}$ if $e \in E_\Lambda^+]$
- $$\mu_{\Lambda,N,\beta}(Q) = Z_{\Lambda,N,\beta}^{-1} \left(\prod_{p \in \mathcal{P}_\Lambda} \exp\{N \cdot \beta \cdot \mathrm{Tr}(Q_p)\} \right) \prod_{e \in E_\Lambda^+} dQ_e$$

The measure wants to favor configurations "close to the identity on plaquettes"

LATTICE YANG-MILLS MEASURE

- $s = (\ell_1, \dots, \ell_n)$ = Collection of n loops in Λ STRING
- $W_s(Q) = \prod_{\ell \in s} \mathrm{tr}(Q_\ell)$, where $Q_\ell = Q_{e_1} \cdots Q_{e_k}$ (if $\ell = e_1, \dots, e_k$) WILSON LOOP OBSERVABLES
- $\phi_{\Lambda,N,\beta}(s) = \mathbb{E}[W_s(Q)]$ w.r.t. $\mu_{\Lambda,N,\beta}$. WILSON LOOP EXPECTATIONS



QUESTION: Can we compute $\phi_{\Lambda,N,\beta}(s)$?

[Ideally, when $\Lambda \uparrow \mathbb{Z}^d$, N is fixed, and β is properly rescaled to have a nontrivial limit]

"THM: [Cao, Park, Sheffield 23]": $\phi_{\Lambda,N,\beta}(s)$ = SUM over RANDOM SURFACES $= \sum_S w(S)$

Embedded maps

Definition: An EMBEDDED MAP is a pair $M = (m, \Psi)$ where m is a collection of planar or higher genus maps and

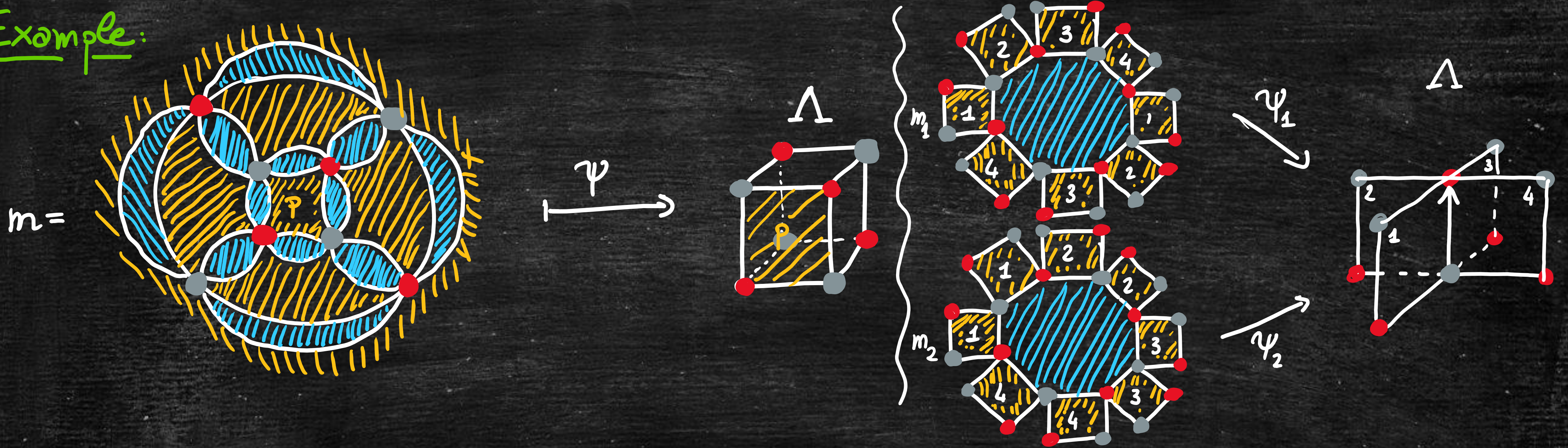
$$\Psi : m \longrightarrow \Lambda$$

is a graph homeomorphism such that:

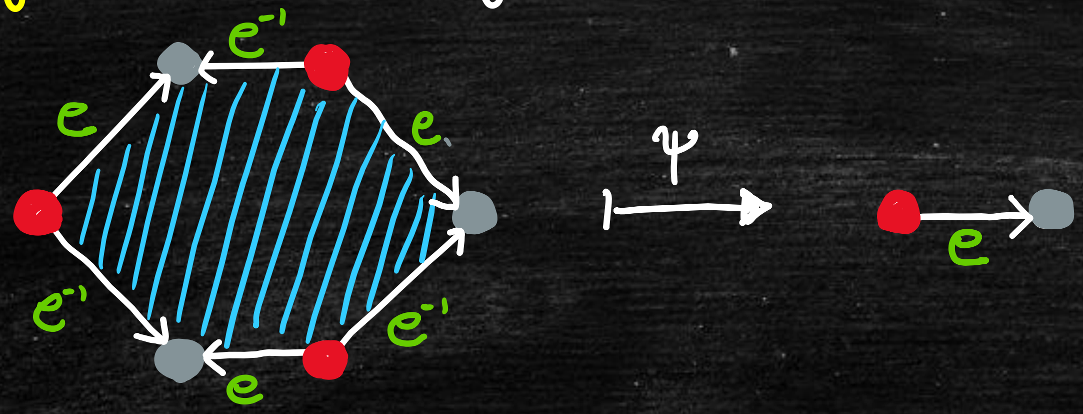
1. The dual graph of each component of m is bipartite (into yellow faces / blue faces)

2. Ψ maps each yellow face isomorphically to a plaquette in $\mathcal{P}_\Lambda$
3. Ψ maps every blue face to a single edge of Λ

Example:



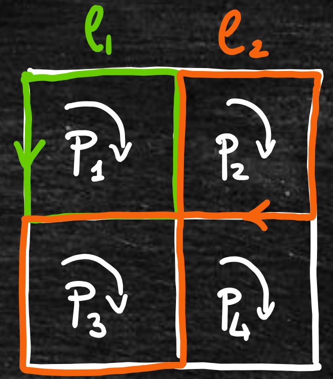
In general, each yellow face has degree 4, while each blue face has EVEN degree



[As a consequence also yellow faces have an orientation]

Note that each edge inherits an orientation from the lattice.

- $s = (\ell_1, \dots, \ell_n)$ = Collection of n loops in Λ STRING



- $K : \mathcal{P}_\Lambda \to \mathbb{N}$ is a PLAQUETTE ASIGNEMENTS

- n_e = # of copies of e in s + $\underbrace{\text{# of copies of e in } K}_{\sum_{p \in \mathcal{P}_\Lambda : e \in p} K(p)}$, $e \in E_\Lambda$

$K(p_1) = 1$ $K(p_1^{-1}) = 2$
$K(p_2) = 0$ $K(p_2^{-1}) = 0$
$K(p_3) = 0$ $K(p_3^{-1}) = 0$
$K(p_4) = 1$ $K(p_4^{-1}) = 0$

- (s, K) is BALANCED if $\forall e \in E_\Lambda^+$, $n_e = n_{e^{-1}}$. Let (s, K) be balanced.

- We denote by $\mathcal{M}(s, K)$ the set of EMBEDDED MAPS M such that
  - # of yellow faces embedded into $p = K(p)$ $\quad \forall p \in \mathcal{P}_\Lambda$
  - Boundaries of M are emdedded to s.

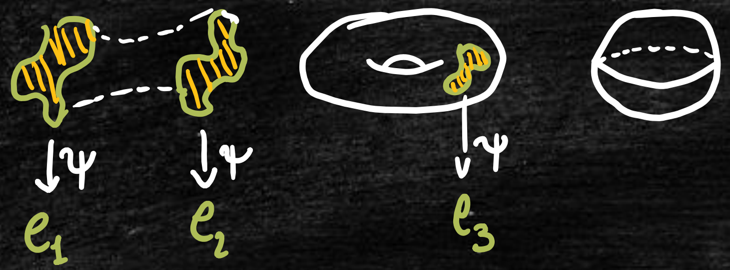
- If (s, K) is not balanced then $\mathcal{M}(s, K) = \phi$.

Reminder:

We denote by $\mathcal{M}(s,K)$ the set of EMBEDDED MAPS M such that

- # of yellow faces embedded onto $p = K(p)$ $\quad \forall p \in \mathcal{P}_\Lambda$
- Boundaries of M are emdedded to s.

INTUITION: Fix $s=(e_1,\dots,e_n)$ and $K:\mathcal{P}_\Lambda \to \mathbb{Z}^d$, then an element $M=(m,\psi)\in\mathcal{M}(s,K)$ looks like this:

$m =$ 

$+$ # of yellow faces mapped onto $p = K(p)$

Remark: The "missing face" along boundaries is always a yellow face

Wilson loops expectations as sum over Random Surfaces

THEOREM [Cao, Park, Sheffield, 23]

Let $s = (\ell_1, \dots, \ell_n)$ be a string. Then

$$\phi_{\Lambda,N,\beta}(s) = Z^{-1}_{\Lambda,N,\beta} \sum_{K : P_\Lambda \to \mathbb{N}}' \ \sum_{M \in \mathcal{M}(s,K)} \beta^{A(M)} \left(\prod_{e \in (s,K)} \underbrace{Wg_N(\mu_e(\psi))} \right) N^{\chi(M)-n}$$

Quite complicated function

where:

- $A(M)$ = # of yellow faces [AREA of M]
- $\chi(M)$ = EULER CHARACTERISTIC of $M = \#V - \#E + \#F$
- $\mu_e(M) = (\lambda_1^e, \dots, \lambda_K^e, \dots)$ is the INTEGER PARTITION of $\frac{\psi^{-1}(e)}{2}$ induced by half of the degrees of the blue faces embedded onto e.

THEOREM [Cao, Park, Sheffield, 23]

Let $s = (\ell_1, \dots, \ell_n)$ be a string. Then

\# of yellow faces

integer partition induced by blue faces mapped onto e

$$\phi_{\Lambda,N,\beta}(s) = Z^{-1}_{\Lambda,N,\beta} \sum_{K:\mathcal{P}_\Lambda \to \mathbb{N}}{}' \ \sum_{M \in \mathcal{M}(s,K)} \beta^{A(M)} \left(\prod_{e \in (s,K)} \underbrace{Wg_N(\mu_e(\psi))}_{\text{Quite complicated function}} \right) N^{\chi(M)-n}$$

Euler-ch.

MAIN ISSUES:

- Wg_N is difficult to study
- $\mathcal{M}(s,K)$ is a quite wild set of maps

HOPE: When N is large, somenthing should be simpler...

Our main results

THEOREM 1 [B., Cao, Shogren-Knaak, 24[+]]:

In any dimension $d \geq 2$, there exists $\beta_0(d) > 0$ such that the following is true.

Let $\Lambda_1, \Lambda_2, \dots$ be any sequence of finite subsets of $\mathbb{Z}^d$ s.t. $\Lambda_N \uparrow \mathbb{Z}^d$.

If $\beta < \beta_0(d)$, then for every string $s = (\ell_1, \dots, \ell_n)$

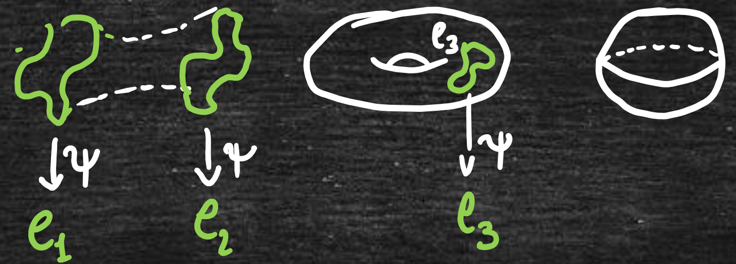
$$\lim_{N \to \infty} \phi_{\Lambda_N, N, \beta}(s) = \prod_{i=1}^{n} \Bigg(\underbrace{\sum_{K : \mathcal{P}_{\mathbb{Z}^d} \to \mathbb{N}} \; \sum_{M \in \mathcal{NPM}(\ell_i, K)} \beta^{A(M)} \, w_\infty(M)}_{= \phi(\ell_i)} \Bigg) =: \phi(s)$$

where

- $w_\infty(M) = \prod_{b \in BF(M)} w_{\deg(f)/2}$ with $w_i = (-1)^{i-1} \, \mathrm{Cat}(i-1)$
- $\mathcal{NPM}(\ell, K) :=$ Set of embedded maps $(m, \psi) \in \mathcal{M}(\ell, K)$ such that m is a (non-separable) planar map with boundary ℓ.

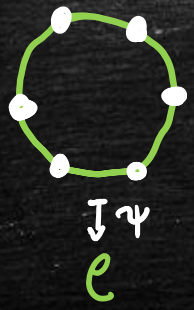
Remarks:

$\boxed{\text{FINITE } N}$

$m =$ 

$+$ # of yellow faces mapped onto $p = K(p)$

$\boxed{N = \infty}$

$m =$ 

$=$ Planar map with the topology of the disk whose boundary is embedded onto ℓ $+$ # of yellow faces mapped onto $p = K(p)$

- The weight of a map is SIMPLE: $\beta^{A(M)} \times \prod_{f \in BF(M)} w_{\frac{\deg(f)}{2}}$

  where $w_i = (-1)^{i-1} \cdot \mathrm{Cat}(i-1)$

- These $(w_i)_i$ are the unique solution to $\begin{cases} w_1 = 1 \\ w_n = -\sum_{i=1}^{n-1} w_i w_{n-i} \end{cases}$

THEOREM 2 [B., Cao, Shogren-Knaak, 24$^+$]:

In dimension $d=2$, for all $\beta \leq \beta_0(2)$ and for any simple loop ℓ we have that

$$\phi(\ell) = \beta^{A(\ell)},$$

where $A(\ell)$ = # of pos. oriented plaquettes contained in ℓ.

IMPORTANT REMARKS :

Analogue results were first obtained for SO(N)-YM (instead of U(N)-YM) by:

- Chatterjee '19 [Large N-factorization]
- Basu-Ganguly '18 [Wilson Loops in $d=2$]

OUR MAIN CONTRIBUTIONS :

- The Random Surface point-of-view gives a new graphical way of doing computations using a sort of PEELING PROCESS.
- The "back-tracking" condition is a NEW important tool
- We get several simplifications in the proofs thanks to the fact that we can work with a fixed plaquette asignement K.

HOPE : These new TOOLS will help us to say something new when

$N<\infty$ / $d>2$ / $d=2$ + β large

THEOREM 2 [B., Cao, Shogren-Knaak, 24^+]:

In dimension $d=2$, for all $\beta \le \beta_0(2)$ and for any simple loop ℓ we have that

$$\phi(\ell) = \beta^{A(\ell)},$$

where $A(\ell)$ = # of pos. oriented plaquettes contained in ℓ.

Comments:

- The Basu-Ganguly proof is elementary only when ℓ is a plaquette.
- In general, it is quite easy to understand where the $\beta^{A(\ell)}$ factor is coming from. The real difficulty is to show that "all the rest" is zero.

MORAL: We need to understand very well certain types of "CANCELLATIONS"

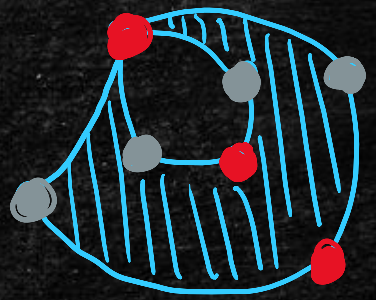
- Our work makes explicit several CANCELLATIONS $\Longrightarrow$ All the proof is elementary!

Main ideas for Theorem 1

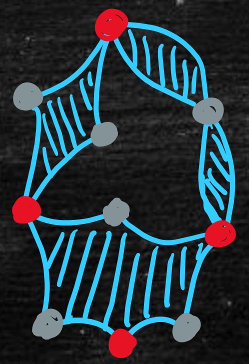
Def: An embedded map $(m, \Psi) \in \mathcal{M}(\mathcal{C}, \mathcal{K})$ is **NON-SEPARABLE** if

1) Every blue face has distinct vertices;

2) There are NO loops of blue faces corresponding to the same edge $e \in E_\Delta$.



NO!



All faces mapped to e

NO!

STEP 1: Guess the formula for a loop ℓ: $\phi(\ell) = \sum_{K : \mathcal{P}_{\mathbb{Z}^d} \to \mathbb{N}} \sum_{M \in \mathcal{NPM}(\ell, K)} \beta^{A(M)} w_\infty(M)$

How do we guessed the formula?

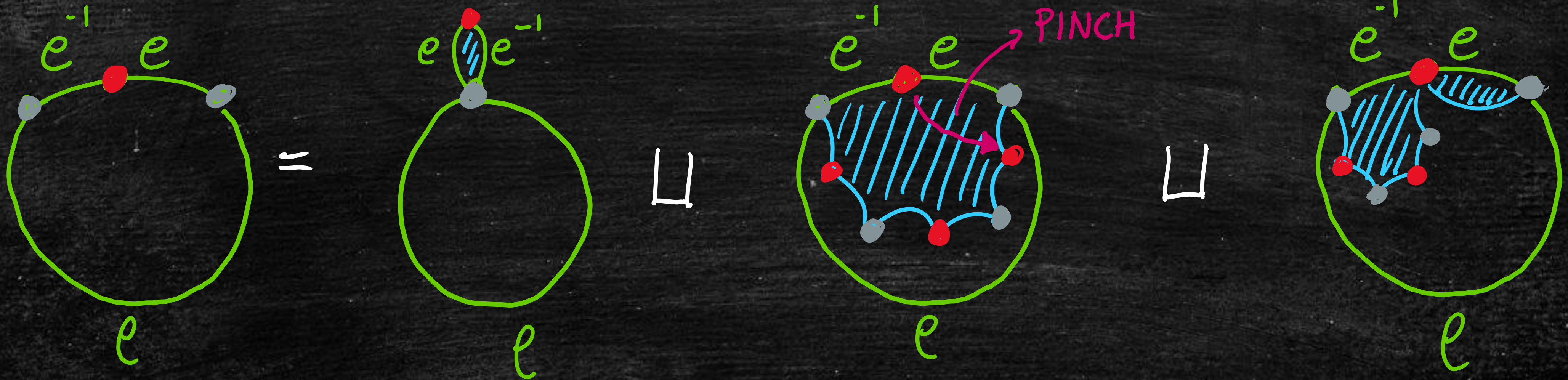
$$\boxed{Z^{-1}_{\Lambda,N,\beta} \sum_{K : \mathcal{P}_\Lambda \to \mathbb{N}}' \sum_{M \in \mathcal{M}(s,K)} \beta^{A(M)} \left(\prod_{e \in \mathcal{E}(s,K)} Wg_N(\mu_e(\gamma)) \right) N^{\chi(M)-n}}$$

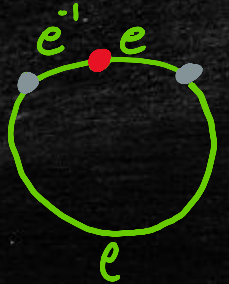
- The limiting expression for the weights w_∞ comes from known results for the limits of Wg_N. In particular, the weights $Wg_N(\cdot)$ factor in the limit along the components of the partition $\Rightarrow$ We can factor out a copy of $Z_{\Lambda,N,\beta}$
  $\Rightarrow$ We can restrict our sum to maps where each component has a boundary mapped to one of the loops.
- The fact that the maps need to be planar is a simple consequence of the factor $N^{\chi(M)-n}$.
- The NON-SEPARABLE condition is NOT OBVIOUS at all!

IDEA: We must be able to cancel back-trackings! That is,

$$\phi(e^{-1}e\ell) = \phi(\ell)$$

If we restrict to our set of maps then



But $w_\infty\Big($  $\Big) = w_\infty\Big($  $\Big) +$ ZERO, because $W_n = -\sum_{i=1}^{n-1} W_i W_{n-i}$

GOAL: Show that when $s = \ell$

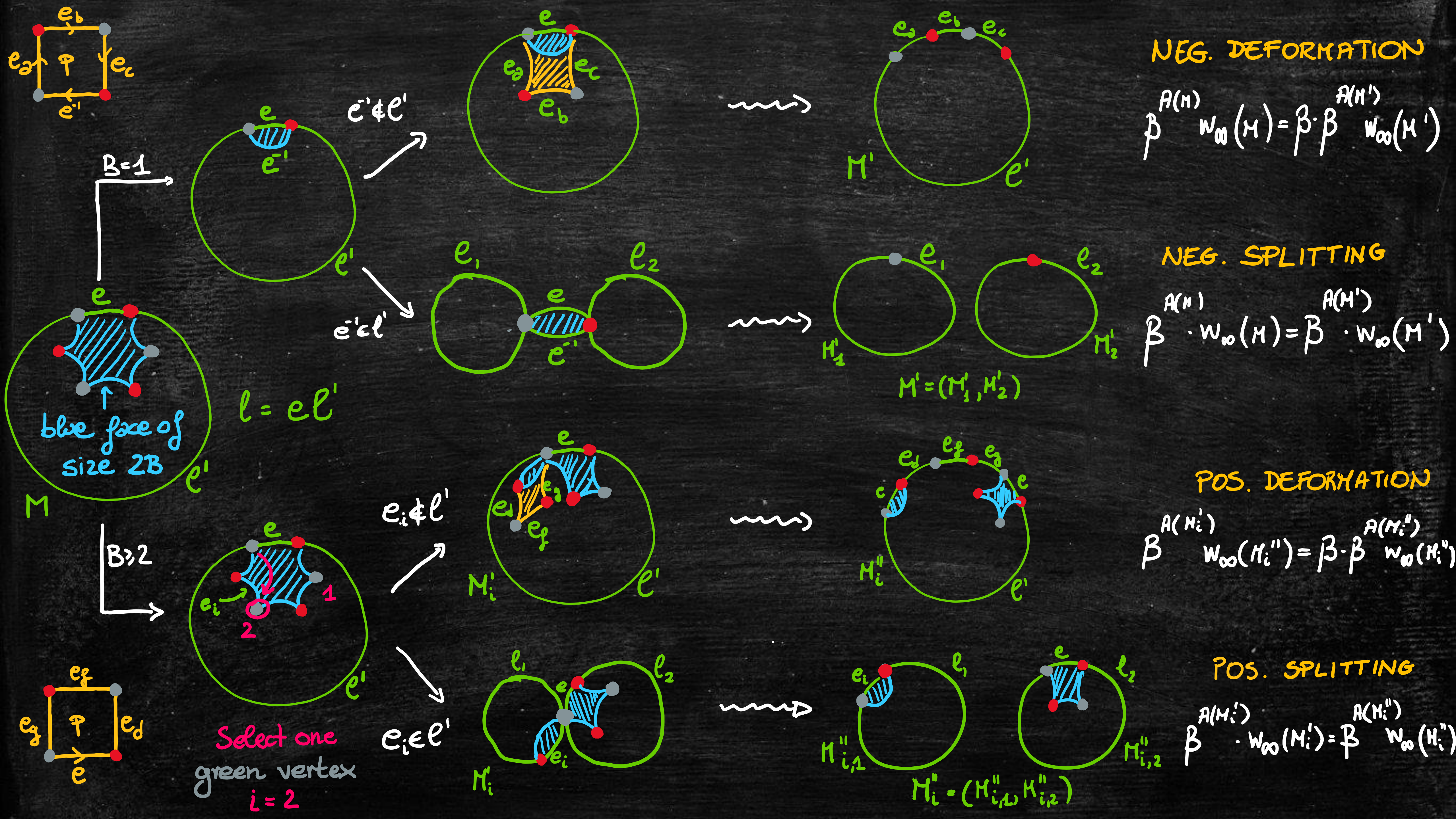
$$\lim_{N\to\infty} \phi_{\Lambda_N, N, \beta}(\ell) = \sum_{K: \mathcal{P}_{\mathbb{Z}^d} \to \mathbb{N}} \underbrace{\sum_{M \in \mathcal{NPM}(\ell, K)} \beta^{A(M)} w_\infty(M)}_{= \phi^K(\ell)} =: \phi(\ell)$$

KEY TOOL:

Show that $\phi^K(\ell)$ satisfies a certain recursive relation, called

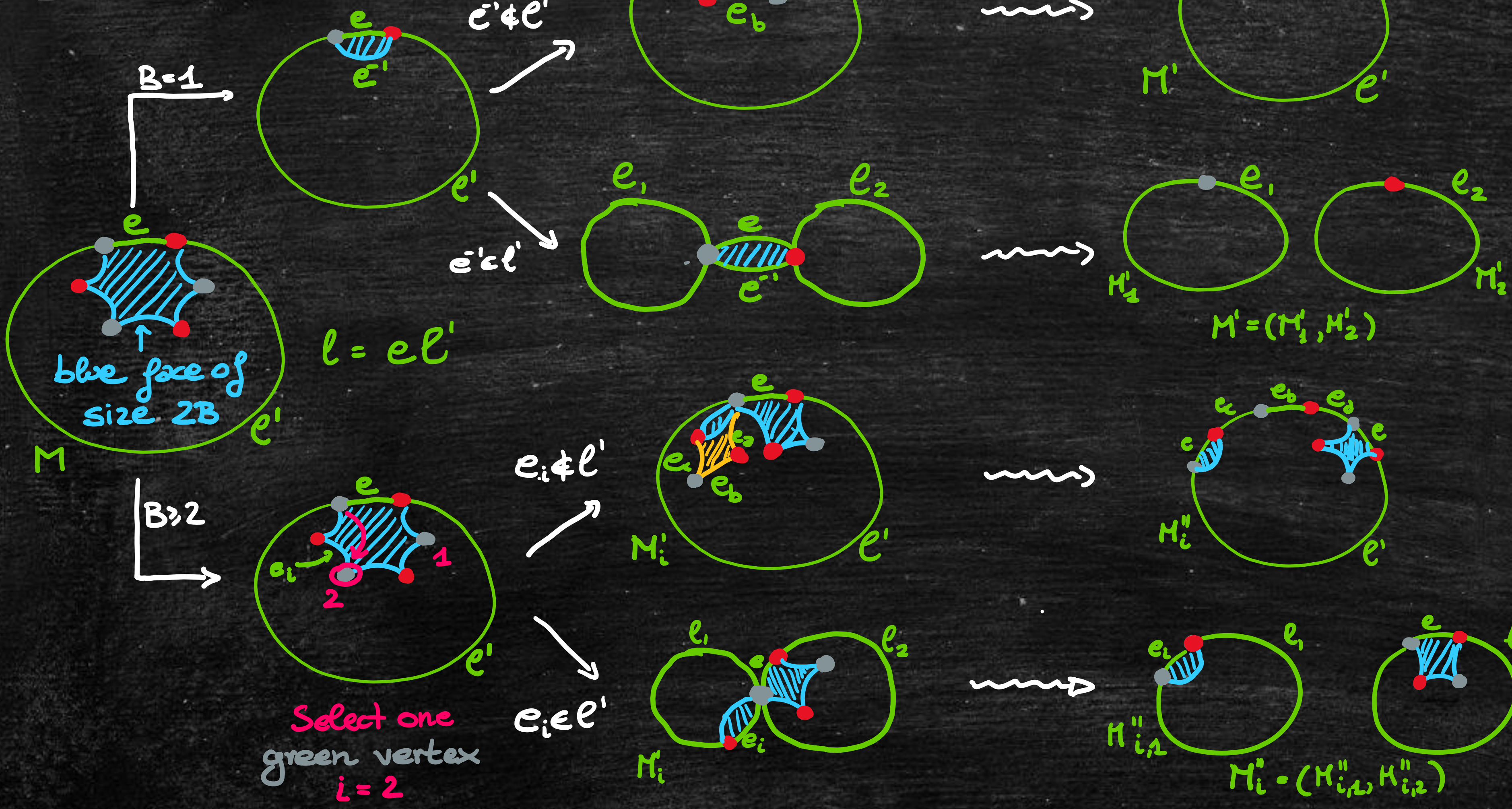
THE MASTER LOOP EQUATION

Indeed, it is simple to show that such equation has a unique solution when β is small & that every subsequential limit of $\phi_{\Lambda_N, N, \beta}(\ell)$ satisfies the equation.



NEG. DEFORMATION
β^{A(M)} W∞(M) = β·β^{A(M')} W∞(M')
NEG. SPLITTING
β^{A(M)} · W∞(M) = β^{A(M')} · W∞(M')
POS. DEFORMATION
β^{A(M'_i)} W∞(M''_i) = β·β^{A(M''_i)} W∞(M''_i)
POS. SPLITTING
β^{A(M'_i)} · W∞(M'_i) = β^{A(M''_i)} W∞(M''_i)
B=1
B≥2
e⁻¹ ∉ ℓ'
e⁻¹ ∈ ℓ'
eᵢ ∉ ℓ'
eᵢ ∈ ℓ'
ℓ = eℓ'
blue face of size 2B
Select one green vertex i = 2
M' = (M'₁, M'₂)
M''ᵢ = (M''ᵢ,₁, M''ᵢ,₂)

K is fixed



NEG. DEFORMATION

$$\beta^{A(M)} W_\infty(M) = \beta \cdot \beta^{A(M')} W_\infty(M')$$

NEG. SPLITTING

$$\beta^{A(M)} \cdot W_\infty(M) = \beta^{A(M')} \cdot W_\infty(M')$$

POS. DEFORMATION

$$\beta^{A(M_i')} W_\infty(M_i'') = \beta \cdot \beta^{A(M_i'')} W_\infty(M_i'')$$

POS. SPLITTING

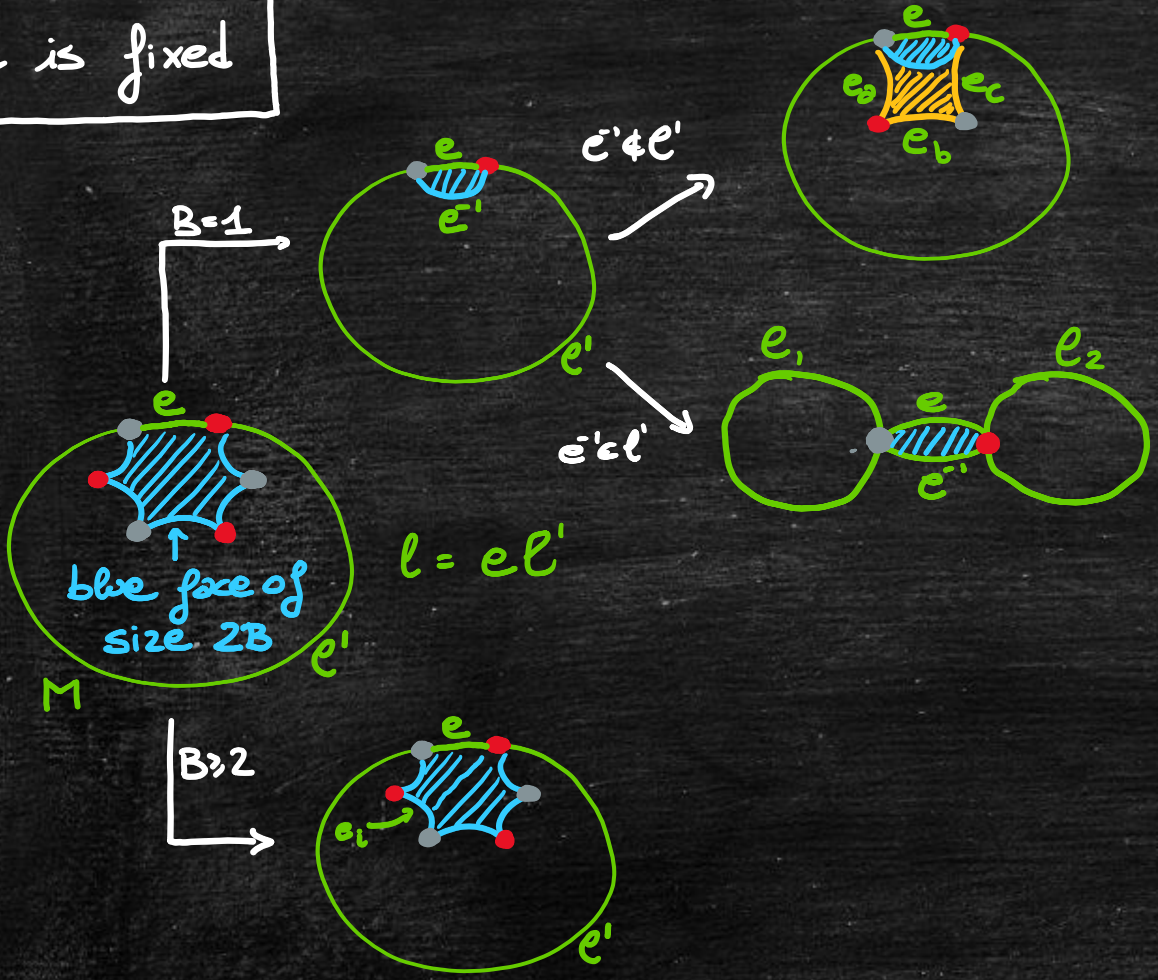
$$\beta^{A(M_i')} \cdot W_\infty(M_i') = \beta^{A(M_i'')} W_\infty(M_i'')$$

We now study $\Phi^K(\ell)$ using the 4 operations above!

REMINDER:

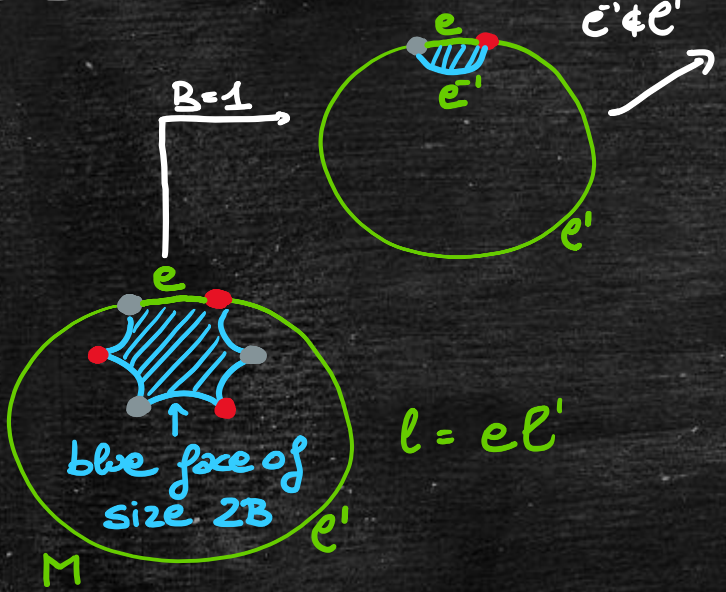
$$\Phi^K(\ell) = \sum_{M \in NPM(\ell,K)} \beta^{A(M)} \cdot W_\infty(M)$$

K is fixed



$$\Phi^K(\ell) = \underbrace{\Phi^K(\ell \mid B=1,\, e' \notin \ell')}_{\text{we now look at this factor}} + \Phi^K(\ell \mid B=1,\, e' \in \ell) + \Phi^K(\ell \mid B \geq 2)$$

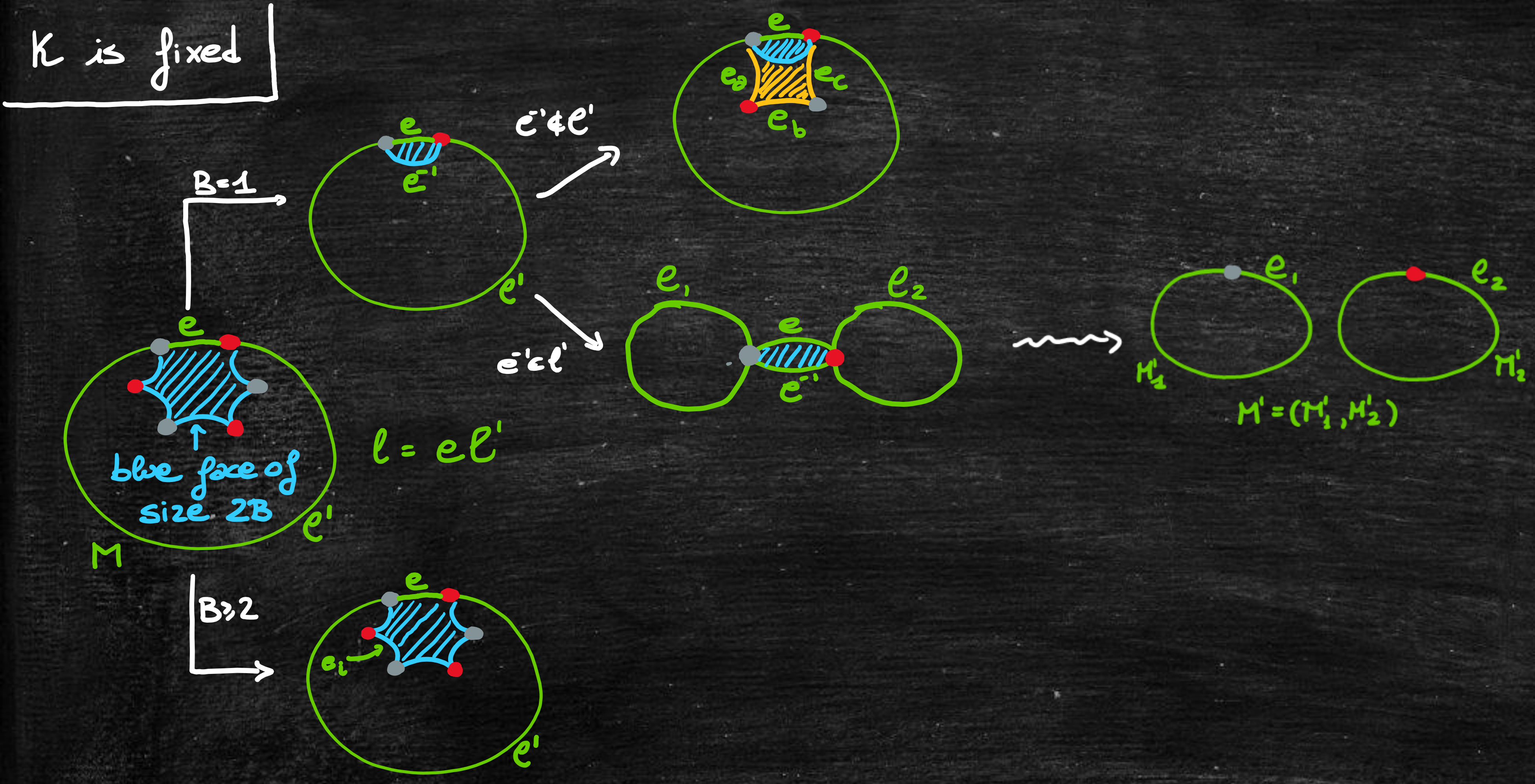
$\boxed{K \text{ is fixed}}$



NEG. DEFORMATION

$$\beta^{A(M)} w_\infty(M) = \beta \cdot \beta^{A(M')} w_\infty(M')$$

$$\Phi^K(\ell \mid B=1, e^{-1}\notin \ell') = \beta \cdot \sum_{\substack{p\in K:\, e^{-1}\in p \\ \parallel \\ (e^{-1}, e_a, e_b, e_c)}} \Phi^{K\setminus p}(e_a e_b e_c \ell')$$



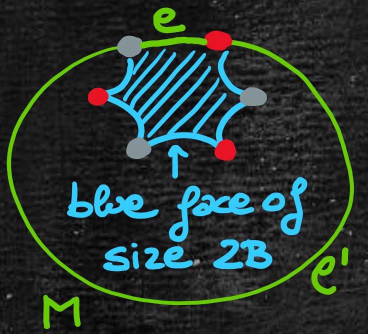
NEG. SPLITTING

$$\beta^{A(M)} \cdot w_\infty(M) = \beta^{A(M')} \cdot w_\infty(M')$$

$$\Phi^K(\ell) = \beta \cdot \sum_{\substack{p\in K:\, e^{-1}\in p \\ \shortparallel \\ (e^{-1}, e_a, e_b, e_c)}} \Phi^{K\setminus p}(e_a e_b e_c \ell') + \sum_{e^{-1}\in \ell'} \Phi^K(\ell_1,\ell_2) + \underbrace{\Phi^K(\ell \mid B\geq 2)}_{?}$$

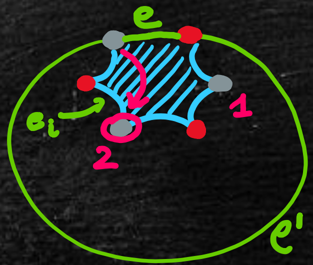
$\boxed{K \text{ is fixed}}$

$B=1$



$\ell = e\ell'$

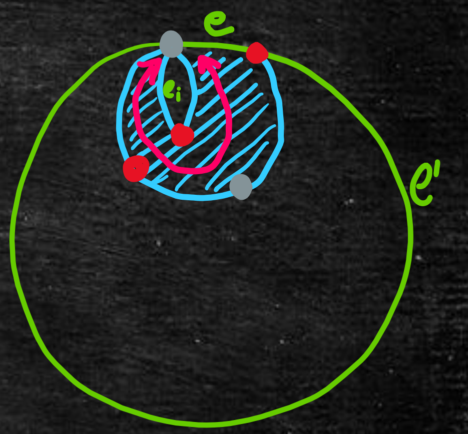
$B \geq 2$



Select one green vertex $i=2$

$$W_b = -\sum_{i=1}^{b-1} w_i \cdot w_{b-i}$$

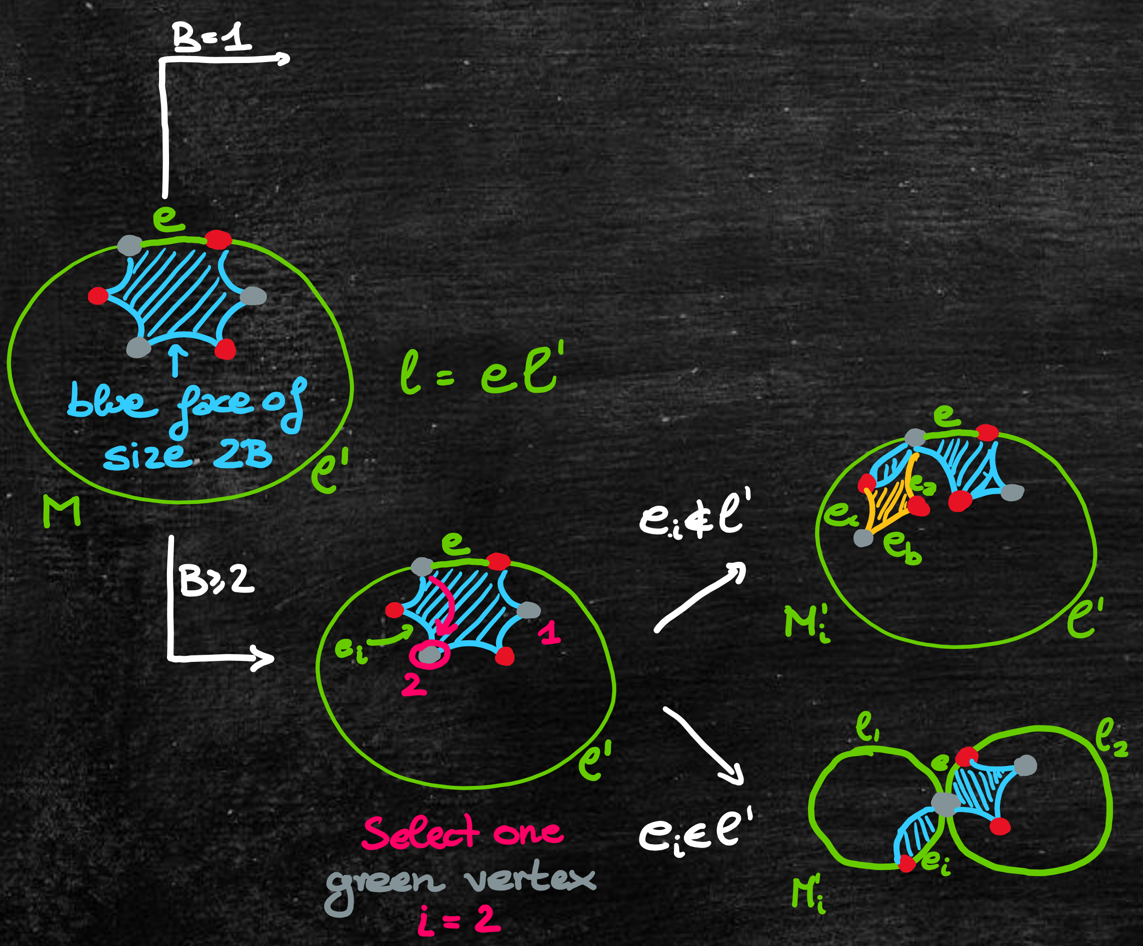
[Here we are using the non-separability]



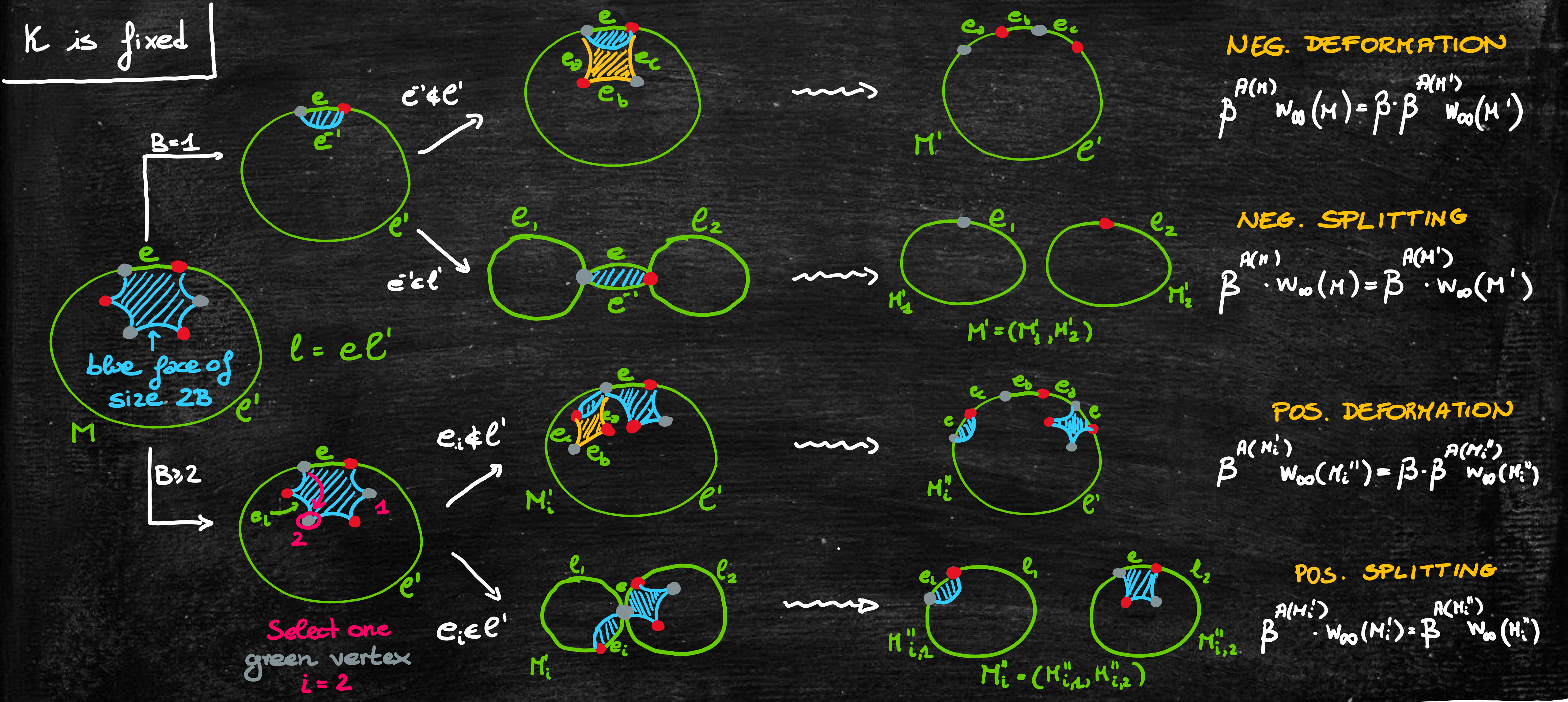
THIS PINCHING WOULD NOT BE POSSIBLE

$$\Phi^K(\ell \mid B \geq 2) = \sum_{b \geq 2} \phi^K(\ell \mid B=b) = -\sum_{b \geq 2} \sum_{i=1}^{b-1} \phi^K(\ell \mid B_1 = i, B_2 = b-i)$$

K is fixed



$$\Phi^K(\ell\,|\,B\geq 2)=\sum_{b\geq 2}\phi^K(\ell\,|\,B=b)= -\sum_{b\geq 2}\sum_{i=1}^{b-1}\phi^K(\ell\,|\,B_1=i,\,B_2=b-i,\,e_i\notin\ell')-\sum_{b\geq 2}\sum_{i=1}^{b-1}\phi^K(\ell\,|\,B_1=i,\,B_2=b-i,\,e_i\in\ell')$$



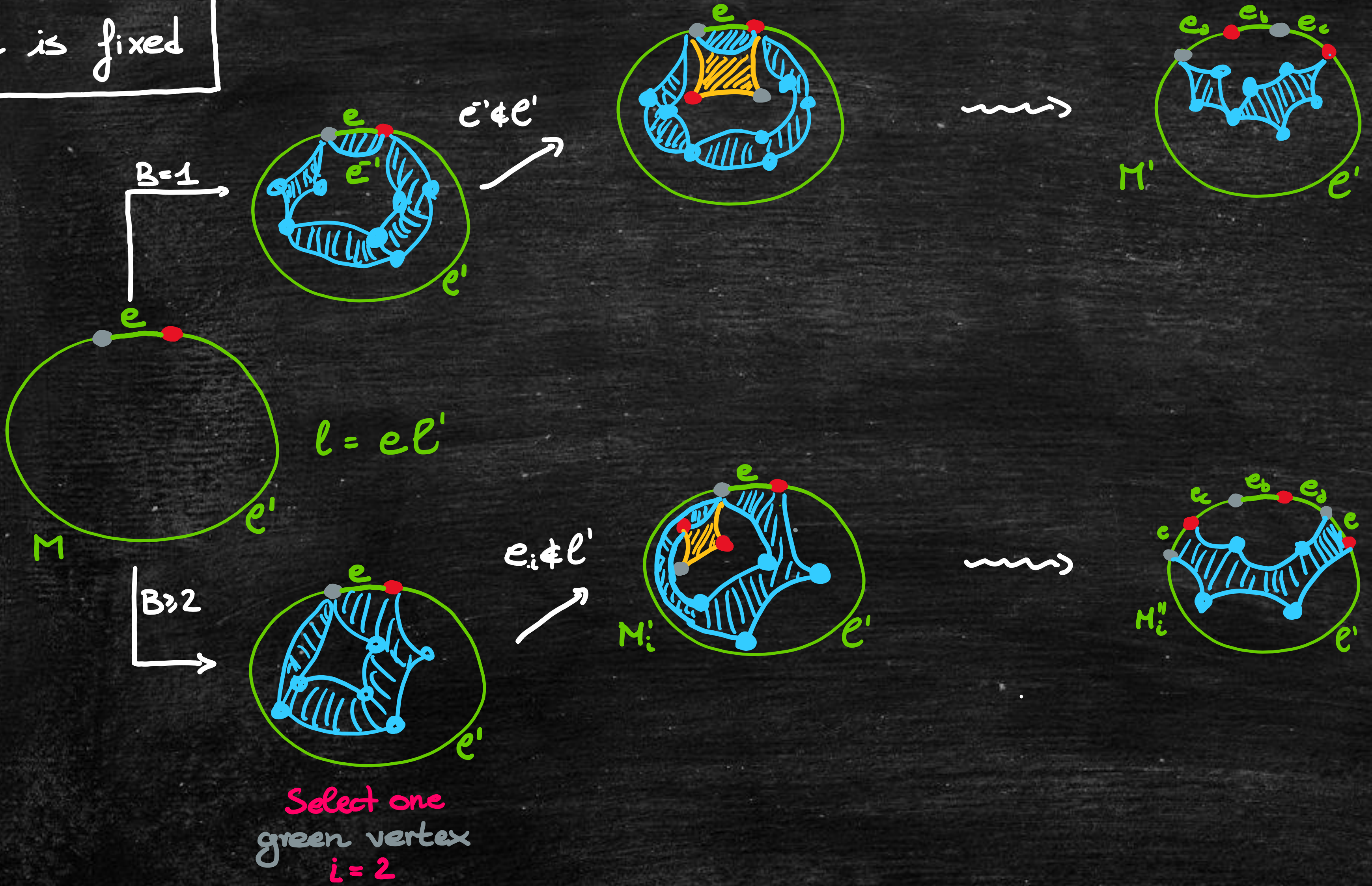
$$\Phi^K(\ell) = \beta \cdot \sum_{\substack{p\in K:\, e^{-1}\in p \\ (e^{-1}, e_a, e_b, e_c)}} \Phi^{K\setminus p}(e_a e_b e_c \ell') + \sum_{e^{-1}\in \ell'} \Phi^K(\ell_1, \ell_2) - \beta \sum_{\substack{p\in K:\, e\in p \\ (e, e_d, e_f, e_g)}} \Phi^{K\setminus p}(e e_d e_f e_g \ell') - \sum_{e\in\ell'} \Phi^K(e\ell_1, e\ell_2)$$

THIS IS THE MASTER LOOP EQ.

K is fixed

THE PREVIOUS ARGUMENT IS WRONG... but NOT too WRONG!

K is fixed



NEG. DEFORMATION

POS. DEFORMATION

The weights of this missing maps sum to ZERO!

THE PREVIOUS ARGUMENT IS WRONG... but NOT too WRONG!

THANK YOU